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A remark on least-squares Galerkin procedures for pseudohyperbolic equations☆

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ABSTRACT

In this paper, we introduce two split least-squares Galerkin finite element procedures for pseudohyperbolic equations arising in the modelling of nerve conduction process. By selecting the least-squares functional properly, the procedures can be split into two sub-procedures, one of which is for the primitive unknown variable and the other is for the flux. The convergence analysis shows that both the two methods yield the approximate solutions with optimal accuracy in $L^2(\Omega)$ norm for u and u_t and $(L^2(\Omega))^2$ norm for the flux σ. Moreover, the two methods get approximate solutions with first-order and second-order accuracy in time increment, respectively. A numerical example is given to show the efficiency of the introduced schemes.

1. Introduction

The disposal of nerve conduction equation is an important biomedical topic and has deep influence on the development of economics and society. In this process, nerve conduction signal u and its variability with respect to time and space can be characterized by a class of pseudohyperbolic equations in mathematics. This kind of model was proposed in [8]. Pao [10–12] and Hasegawa [2] gave some further study and extension for the numerical model. [13,17] have given some results about the uniqueness, existence and asymptotic behavior of solutions for this problem.

We have introduced an elegant theory of the least-squares methods for pseudohyperbolic equations in [6]. The least-squares Galerkin procedure has two typical advantages as follows: it is not subject to the Ladyzhenskaya [7]-Babuska [3]-Brezzi [4] consistency condition, so the choice of approximation spaces becomes flexible, and it results in a symmetric positive definite system.

Recently, in [16,15] a kind of split least-squares Galerkin procedure was constructed for stationary diffusion reaction problems and parabolic problems. In this paper, we apply this idea and give some split least-squares Galerkin procedures to solve the pseudohyperbolic equations. By selecting the least-squares functional properly, the resulting least-squares Galerkin procedures can be split into two symmetric positive definite sub-schemes, one of which is for the unknown variable u_t and the other sub-scheme is for the introduced unknown flux variable σ. The convergence analysis shows that

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the methods yield the approximate solutions with optimal accuracy. Finally, we give a numerical example to testify the efficiency of the split least-squares Galerkin schemes.

The paper is organized as follows. In Section 2 we formulate the split least-squares Galerkin procedures. The convergence theory on the novel algorithms is established in Section 3. In Section 4 we give the numerical experiment.

In this paper we use $W^{k,p}(k \geq 0, 1 \leq p \leq \infty)$ to denote Soboleve spaces [1] defined on Ω with usual norms $\|\cdot\|_{W^{k,p}(\Omega)}$ and $H^k(\Omega) = W^{k,2}(\Omega)$ with norms $\|\cdot\|_k$. For simplicity we also use $L^s(H^k)$ to denote $L^s(0, T; H^k(\Omega))$. The inner product $(\cdot,\cdot)$ is both used for scalar-valued functions and vector-valued functions. Throughout this paper, the symbols K and δ are used to denote a generic constant and a generic small positive constant, respectively.

2. Split least-squares Galerkin finite element procedures

We consider the following initial-boundary value problem of pseudohyperbolic system

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + q(x)u_t = f(x,t), \quad \text{in } \Omega \times J, \\ u(x,t)|_{\Gamma\times J} = 0, \qquad u_t(x,t)|_{\Gamma\times J} = 0, \\ u(x,0) = u_0(x), \quad \text{in } \Omega, \\ u_t(x,0) = w_0(x), \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where $J = (0, T)$ is the time interval, Ω is a bounded polygonal domain in R^d, $1 \leq d \leq 3$, with a Lipschitz continuous boundary Γ. $f = f(x,t)$ is a given function. We shall make the following assumption on the coefficient $q(x)$: there exist positive constants q_*, q^* such that

$$q_* \leq q(x) \leq q^*.$$

We also assume that Ω is H^2-regular: for $f \in L^2(\Omega)$, the solution of the following problem

$$\begin{cases} -\nabla\cdot(\nabla w) = f, \quad \text{in } \Omega, \\ w|_\Gamma = 0, \end{cases}$$

exists and

$$\|w\|_2 \leq C\|f\|.$$

Introduce two function spaces

$$\mathrm{H} = \{\psi \in L^2(\Omega)^2; \operatorname{div}\psi \in L^2(\Omega)\},$$
$$\mathrm{S} = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma\}.$$

Introducing the flux variable $\sigma = -(\nabla u_t + \nabla u)$, we can rewrite problem (2.1) as a first-order system

$$\begin{cases} u_{tt} + \operatorname{div}\sigma(x,t) + q(x)u_t = f(x,t), \quad x \in \Omega,\ 0 < t \leq T, \\ \sigma(x,t) + \nabla u_t + \nabla u = 0, \quad x \in \Omega,\ 0 < t \leq T, \\ u(x,t) = 0, \qquad u_t(x,t) = 0, \quad x \in \Gamma,\ 0 < t \leq T, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = w_0(x), \quad x \in \Omega. \end{cases} \tag{2.2}$$

Since u_t is an important physical parameter in process of nerve conduction, we also need its error estimate. We consider another system equivalent to (2.2) as follows

$$\begin{cases} u_{tt} + \operatorname{div}\sigma(x,t) + q(x)u_t = f(x,t), \quad x \in \Omega,\ 0 < t \leq T, \\ \sigma_t(x,t) + \nabla u_{tt} + \nabla u_t = 0, \quad x \in \Omega,\ 0 < t \leq T, \\ u(x,t) = 0, \qquad u_t(x,t) = 0, \quad x \in \Gamma,\ 0 < t \leq T, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = w_0(x), \quad x \in \Omega. \end{cases} \tag{2.3}$$

Using general transformation $w = u_t$, then (2.3) can be put into the form

$$\begin{cases} w_t + \operatorname{div}\sigma(x,t) + q(x)w = f(x,t), \quad x \in \Omega,\ 0 < t \leq T, \\ \sigma_t(x,t) + \nabla w_t + \nabla w = 0, \quad x \in \Omega,\ 0 < t \leq T, \\ w(x,t) = 0, \quad x \in \Gamma,\ 0 < t \leq T, \\ w(x,0) = w_0(x), \quad x \in \Omega. \end{cases} \tag{2.4}$$

Given a time step $\Delta t = T/N$, where N is a positive integer, we shall approximate the solution at times $t_n = n\Delta t$, $n = 0, 1, \ldots, N$. Let $w^n(x) = w(x, t_n)$ and $w^{n-\frac{1}{2}}(x) = w(x, t_{n-1} + \Delta t/2)$.

By using the difference technique with first-order accuracy in time, we can rewrite system (2.4) as the following semi-discrete system of form **I**: for $n \geq 1$, find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$\begin{cases} \dfrac{w^n - w^{n-1}}{\Delta t} + \operatorname{div}\sigma^n + qw^n = f^n + R_1^n, \quad x \in \Omega, \\ \dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} + \nabla\dfrac{w^n - w^{n-1}}{\Delta t} + \nabla w^n = R_2^n, \quad x \in \Omega, \\ w^n(x) = 0, \quad x \in \Gamma, \\ w^0(x) = w_0(x), \quad x \in \Omega, \end{cases} \tag{2.5}$$

where

$$R_1^n = D_t w^n - w_t^n = O(\Delta t),$$
$$R_2^n = D_t \sigma^n - \sigma_t^n + \nabla(D_t w^n - w_t^n) = O(\Delta t),$$
$$D_t w^n = \frac{w^n - w^{n-1}}{\Delta t}.$$

Equivalently from (2.5), we have that

$$\begin{cases} c^{-\frac{1}{2}}\left[c w^n + \Delta t \operatorname{div} \sigma^n - (w^{n-1} + \Delta t f^n + \Delta t R_1^n)\right] = 0, \quad x \in \Omega, \\ A^{-\frac{1}{2}}\left[\sigma^n + A\nabla w^n - (\sigma^{n-1} + \nabla w^{n-1} + \Delta t R_2^n)\right] = 0, \quad x \in \Omega, \\ w^n(x) = 0, \quad x \in \Gamma, \\ w^0(x) = w_0(x), \quad x \in \Omega, \end{cases} \tag{2.6}$$

where

$$c(x) = 1 + \Delta t q(x), \qquad A = 1 + \Delta t.$$

Define the minimization functional J_1^n as

$$J_1^n(\psi, v) = \|c^{-\frac{1}{2}}[cv + \Delta t \operatorname{div} \psi - (w^{n-1} + \Delta t f^n + \Delta t R_1^n)]\|^2 + \Delta t \|A^{-\frac{1}{2}}[\psi + A\nabla v - (\sigma^{n-1} + \nabla w^{n-1} + \Delta t R_2^n)]\|^2.$$

Then the least-squares minimization corresponding to (2.6) is: find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$J_1^n(\sigma^n, w^n) = \min_{\psi \in H, v \in S} J_1^n(\psi, v). \tag{2.7}$$

The weak formulation of (2.7) is: find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$a_n((\sigma^n, w^n), (\psi, v)) = \left(c^{-1}(w^{n-1} + \Delta t f^n + \Delta t R_1^n), cv + \Delta t \operatorname{div} \psi\right) + \Delta t \left(\left(A^{-1}(\sigma^{n-1} + \nabla w^{n-1} + \Delta t R_2^n)\right), \psi + A\nabla v\right), \quad \forall (\psi, v) \in H \times S. \tag{2.8}$$

Here the bilinear form a_n is defined as

$$a_n((\sigma, w), (\psi, v)) = \left(c^{-1}(cw + \Delta t \operatorname{div} \sigma), cv + \Delta t \operatorname{div} \psi\right) + \Delta t \left(A^{-1}(\sigma + A\nabla w), \psi + A\nabla v\right). \tag{2.9}$$

Now we give another weak form of the problem. By using the difference technique with second-order accuracy in time, we can rewrite system (2.4) as the following semi-discrete system of form **II:** for $n \geq 1$, find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$\begin{cases} \dfrac{w^n - w^{n-1}}{\Delta t} + \operatorname{div} \hat{\sigma}^{n-\frac{1}{2}} + q\hat{w}^{n-\frac{1}{2}} = f^{n-\frac{1}{2}} + R_3^n, \quad x \in \Omega, \\ \dfrac{\sigma^n - \sigma^{n-1}}{\Delta t} + \nabla \dfrac{w^n - w^{n-1}}{\Delta t} + \nabla \hat{w}^{n-\frac{1}{2}} = R_4^n, \quad x \in \Omega, \\ w^n(x) = 0, \quad x \in \Gamma, \\ w^0(x) = w_0(x), \quad x \in \Omega, \end{cases} \tag{2.10}$$

where

$$\hat{w}^{n-1/2} = \frac{(w^n + w^{n-1})}{2}, \qquad \hat{\sigma}^{n-1/2} = \frac{(\sigma^n + \sigma^{n-1})}{2},$$
$$R_3^n = (D_t w^n - w_t^{n-\frac{1}{2}}) + \operatorname{div}(\hat{\sigma}^{n-\frac{1}{2}} - \sigma^{n-\frac{1}{2}}) + q(\hat{w}^{n-\frac{1}{2}} - w^{n-1/2}) = O(\Delta t^2),$$
$$R_4^n = (D_t \sigma^n - \sigma_t^{n-\frac{1}{2}}) + \nabla(D_t w^n - w_t^{n-\frac{1}{2}}) + \nabla(\hat{w}^{n-\frac{1}{2}} - w^{n-\frac{1}{2}}) = O(\Delta t^2),$$

or equivalently

$$\begin{cases} \tilde{c}^{-\frac{1}{2}}\left[\tilde{c}w^n + \dfrac{\Delta t}{2}\operatorname{div} \sigma^n - \left((\tilde{c} - \Delta t q)w^{n-1} - \dfrac{\Delta t}{2}\operatorname{div} \sigma^{n-1} + \Delta t f^{n-\frac{1}{2}} + \Delta t R_3^n\right)\right] = 0, \quad x \in \Omega, \\ \tilde{A}^{-\frac{1}{2}}\left[\sigma^n + \tilde{A}\nabla w^n - \left(\sigma^{n-1} + (\tilde{A} - \Delta t)\nabla w^{n-1} + \Delta t R_4^n\right)\right] = 0, \quad x \in \Omega, \\ w^n(x) = 0, \quad x \in \Gamma, \\ w^0(x) = w_0(x), \quad x \in \Omega, \end{cases} \tag{2.11}$$

where

$$\tilde{c}(x) = 1 + \frac{\Delta t}{2} q(x), \qquad \tilde{A} = 1 + \frac{\Delta t}{2}.$$

Define the minimization functional J_2^n as

$$J_2^n(\psi, v) = \left\| \tilde{c}^{-\frac{1}{2}} \left[\tilde{c}v + \frac{\Delta t}{2}\mathrm{div}\,\psi - \left((\tilde{c} - \Delta tq)w^{n-1} - \frac{\Delta t}{2}\mathrm{div}\,\sigma^{n-1} + \Delta tf^{n-\frac{1}{2}} + \Delta tR_3^n \right) \right] \right\|^2 + \frac{\Delta t}{2} \|\tilde{A}^{-\frac{1}{2}}[\psi + \tilde{A}\nabla v - (\sigma^{n-1} + (\tilde{A} - \Delta t)\nabla w^{n-1} + \Delta tR_4^n)]\|^2.$$

Then the least-squares minimization corresponding to (2.11) is: find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$J_2^n(\sigma^n, w^n) = \min_{\psi \in H, v \in S} J_2^n(\psi, v). \tag{2.12}$$

The weak formulation of (2.12) is: find $(\sigma^n, w^n) \in \mathrm{H} \times \mathrm{S}$ such that

$$b_n((\sigma^n, w^n), (\psi, v)) = \left(\tilde{c}^{-1} \left((\tilde{c} - \Delta tq)w^{n-1} - \frac{\Delta t}{2}\mathrm{div}\,\sigma^{n-1} + \Delta tf^{n-\frac{1}{2}} + \Delta tR_3^n \right), \tilde{c}v + \frac{\Delta t}{2}\mathrm{div}\,\psi \right) + \frac{\Delta t}{2} \left(\tilde{A}^{-1}(\sigma^{n-1} + (\tilde{A} - \Delta t)\nabla w^{n-1} + \Delta tR_4^n), \psi + \tilde{A}\nabla v \right), \quad \forall (\psi, v) \in H \times S, \tag{2.13}$$

where we define another bilinear form b_n as

$$b_n((\sigma, w), (\psi, v)) = \left(\tilde{c}^{-1} \left(\tilde{c}w + \frac{\Delta t}{2}\mathrm{div}\,\sigma \right), \tilde{c}v + \frac{\Delta t}{2}\mathrm{div}\,\psi \right) + \frac{\Delta t}{2} \left(\tilde{A}^{-1}(\sigma + \tilde{A}\nabla w), \psi + \tilde{A}\nabla v \right). \tag{2.14}$$

We will give the two split least-squares Galerkin procedures based on (2.8) and (2.13). Let T_{h_σ} and T_{h_u} be two families of finite element partitions of the domain Ω, which are identical or not. h_σ and h_u are mesh parameters, respectively. The corresponding finite element spaces $H_{h_\sigma} \subset H$ and $S_{h_u} \subset S$ have the following approximate properties: there exist integers $k_1 \geq k \geq 0,\ l \geq 1$ such that

$$\begin{cases} \inf\limits_{\omega_h \in H_{h_\sigma}} \|\omega - \omega_h\| \leq Kh_\sigma^{k+1}\|\omega\|_{k+1}, \\ \inf\limits_{\omega_h \in H_{h_\sigma}} \|\mathrm{div}\,(\omega - \omega_h)\| \leq Kh_\sigma^{k_1}\|\omega\|_{k_1+1}, \\ \inf\limits_{v_h \in S_{h_u}} \{\|v - v_h\| + h_u\|\nabla(v - v_h)\|\} \leq Kh_u^{l+1}\|v\|_{l+1}, \end{cases} \tag{2.15}$$

where $k_1 = k + 1$ in the case that the space H_{h_σ} is any one of Raviart–Thomas mixed elements [14] and Nedelec mixed elements [9] and $k_1 = k \geq 1$ in the case that the space H_{h_σ} is the C^0-elements [5].

We select the initial approximation such that

$$\begin{cases} \|u_0 - u_h^0\|_s \leq Kh_u^{l+1-s}\|u_0\|_{l+1}, \quad s = 0, 1, \\ \|w_0 - w_h^0\|_s \leq Kh_u^{l+1-s}\|w_0\|_{l+1}, \quad s = 0, 1, \\ \|\sigma_0 - \sigma_h^0\| \leq Kh_\sigma^{k+1}\|\sigma_0\|_{k+1}. \end{cases} \tag{2.16}$$

Omitting the time truncation error terms in (2.8), we define the first least-squares Galerkin finite element procedure.

Scheme **I**. With the initial approximation $\sigma_h^0 \in H_{h_\sigma}$, $w_h^0 = Rw^0 \in S_{h_u}$ (which is defined in (3.2)), for $n = 1, 2 \cdots N$, we seek $(\sigma_h^n, w_h^n) \in H_{h_\sigma} \times S_{h_u}$ such that

$$a_n((\sigma_h^n, w_h^n), (\psi_h, v_h)) = \left(c^{-1}(w_h^{n-1} + \Delta tf^n), cv_h + \Delta t\mathrm{div}\,\psi_h\right) + \Delta t(A^{-1}(\sigma_h^{n-1} + \nabla w_h^{n-1}), \psi_h + A\nabla v_h), \quad \forall (\psi_h, v_h) \in H_{h_\sigma} \times S_{h_u}. \tag{2.17}$$

Similarly, based on (2.13), we give the second least-squares Galerkin finite element procedure.

Scheme **II**. With the initial approximation $\sigma_h^0 \in H_{h_\sigma}$, $w_h^0 = Rw^0 \in S_{h_u}$, for $n = 1, 2 \cdots N$, we seek $(\sigma_h^n, w_h^n) \in H_{h_\sigma} \times S_{h_u}$ such that

$$b_n((\sigma_h^n, w_h^n), (\psi_h, v_h)) = \left(\tilde{c}^{-1} \left((\tilde{c} - \Delta tq)w_h^{n-1} - \frac{\Delta t}{2}\mathrm{div}\,\sigma_h^{n-1} + \Delta tf^{n-\frac{1}{2}} \right), \tilde{c}v_h + \frac{\Delta t}{2}\mathrm{div}\,\psi_h \right) + \frac{\Delta t}{2} \left(\tilde{A}^{-1}(\sigma_h^{n-1} + (\tilde{A} - \Delta t)\nabla w_h^{n-1}), \psi_h + \tilde{A}\nabla v_h \right), \quad \forall (\psi_h, v_h) \in H_{h_\sigma} \times S_{h_u}. \tag{2.18}$$

Now we discuss the bilinear forms a_n and b_n in the following lemma, which leads to decoupled systems.

Lemma 2.1. *For any $(\sigma, w), (\psi, v) \in H \times S$, we have that*

$$a_n((\sigma, w), (\psi, v)) = (cw, v) + \Delta t^2(c^{-1}\mathrm{div}\,\sigma, \mathrm{div}\,\psi) + \Delta t(A^{-1}\sigma, \psi) + \Delta t(A\nabla w, \nabla v), \tag{2.19}$$

$$b_n((\sigma, w), (\psi, v)) = (\tilde{c}w, v) + \frac{\Delta t^2}{4}(\tilde{c}^{-1}\mathrm{div}\,\sigma, \mathrm{div}\,\psi) + \frac{\Delta t}{2}(\tilde{A}^{-1}\sigma, \psi) + \frac{\Delta t}{2}(\tilde{A}\nabla w, \nabla v). \tag{2.20}$$

Proof. A direct calculation shows that

$$a_n((\sigma,w),(\psi,v)) = (cw,v) + \Delta t^2(c^{-1}\mathrm{div}\,\sigma, \mathrm{div}\,\psi) + \Delta t(A^{-1}\sigma,\psi) + \Delta t(A\nabla w,\nabla v) + \Delta t[(w,\mathrm{div}\,\psi) + (\mathrm{div}\,\sigma, v) + (\sigma,\nabla v) + (\nabla w,\psi)].$$

By applying Green's formula, we have

$$(w,\mathrm{div}\,\psi) + (\mathrm{div}\,\sigma, v) + (\sigma,\nabla v) + (\nabla w,\psi) = 0,$$

which completes the proof of (2.19). Similarly, we can prove the second one. #

Based on this lemma, we give the split equivalent forms of Scheme **I** and Scheme **II**.

Remark 2.1. Let $\psi_h = 0$ and $v_h = 0$ in (2.17), alternatively. Using Lemma 2.1, we have the equivalent form of Scheme **I**: find $(\sigma_h^n, w_h^n) \in H_{h_\sigma} \times S_{h_u}$ such that

$$(cw_h^n, v_h) + \Delta t(A\nabla w_h^n, \nabla v_h) = (w_h^{n-1}, v_h) + \Delta t(f^n, v_h) + \Delta t(\sigma_h^{n-1}, \nabla v_h) + \Delta t(\nabla w_h^{n-1}, \nabla v_h), \quad \forall v_h \in S_{h_u}, \tag{2.21}$$

$$\begin{aligned}(A^{-1}\sigma_h^n, \psi_h) + \Delta t(c^{-1}\mathrm{div}\,\sigma_h^n, \mathrm{div}\,\psi_h) &= \left(c^{-1}(w_h^{n-1} + \Delta t f^n), \mathrm{div}\,\psi_h\right) + \left(A^{-1}(\sigma_h^{n-1} + \nabla w_h^{n-1}), \psi_h\right)\\ &= (A^{-1}\sigma_h^{n-1}, \psi_h) - \Delta t\left((c^{-1}q - A^{-1})w_h^{n-1}, \mathrm{div}\,\psi_h\right)\\ &\quad + \Delta t(c^{-1}f^n, \mathrm{div}\,\psi_h), \quad \forall \psi_h \in H_{h_\sigma}.\end{aligned} \tag{2.22}$$

Remark 2.2. Let $\psi_h = 0$ and $v_h = 0$ in (2.18), alternatively. Using Lemma 2.1, we have the equivalent form of Scheme **II:** find $(\sigma_h^n, w_h^n) \in H_{h_\sigma} \times S_{h_u}$ such that

$$\begin{aligned}(\tilde{c}w_h^n, v_h) + \frac{\Delta t}{2}(\tilde{A}\nabla w_h^n, \nabla v_h) &= \left((\tilde{c} - \Delta t q)w_h^{n-1}, v_h\right) + \Delta t(f^{n-\frac{1}{2}}, v_h)\\ &\quad + \Delta t(\sigma_h^{n-1}, \nabla v_h) + \frac{\Delta t}{2}((\tilde{A} - \Delta t)\nabla w_h^{n-1}, \nabla v_h), \quad \forall v_h \in S_{h_u},\end{aligned} \tag{2.23}$$

$$\begin{aligned}(\tilde{A}^{-1}\sigma_h^n, \psi_h) + \frac{\Delta t}{2}(\tilde{c}^{-1}\mathrm{div}\,\sigma_h^n, \mathrm{div}\,\psi_h) &= \left(\tilde{c}^{-1}((\tilde{c} - \Delta t q)w_h^{n-1} - \frac{\Delta t}{2}\mathrm{div}\,\sigma_h^{n-1} + \Delta t f^{n-\frac{1}{2}}), \mathrm{div}\,\psi_h\right)\\ &\quad + \left(\tilde{A}^{-1}(\sigma_h^{n-1} + (\tilde{A} - \Delta t)\nabla w_h^{n-1}), \psi_h\right)\\ &= (\tilde{A}^{-1}\sigma_h^{n-1}, \psi_h) - \frac{\Delta t}{2}(\tilde{c}^{-1}\mathrm{div}\,\sigma_h^{n-1}, \mathrm{div}\,\psi_h) + \Delta t(\tilde{c}^{-1}f^{n-\frac{1}{2}}, \mathrm{div}\,\psi_h)\\ &\quad - \Delta t\left((\tilde{c}^{-1}q - \tilde{A}^{-1})w_h^{n-1}, \mathrm{div}\,\psi_h\right), \quad \forall \psi_h \in H_{h_\sigma}.\end{aligned} \tag{2.24}$$

From these results we see that by selecting the least-squares functional properly, Scheme **I** and Scheme **II** can be split into two sub-procedures, one of which is for the primitive unknown w_h and the other is for the flux σ_h.

Lemma 2.1 also tells us that the bilinear forms $a_n(\cdot,\cdot)$ and $b_n(\cdot,\cdot)$ are continuous and positive definite in $H \times S$. So it follows from Lax-Milgram theorem that Scheme **I** and Scheme **II** both have a unique solution.

3. Convergence analysis

In this section, we analyze the convergence of the two procedures. For this purpose we introduce some project operators first.

From the approximate property of finite element spaces we know that for a given $\sigma \in H(\mathrm{div}\,;\Omega) \cap H^{k_1+1}(\Omega)$ there exists a vector-valued function $\mathrm{Q}\sigma \in H_{h_\sigma}$ such that

$$\begin{cases}\|\sigma - \mathrm{Q}\sigma\| \le Kh_\sigma^{k+1}\|\sigma\|_{k+1},\\ \|\mathrm{div}\,(\sigma - \mathrm{Q}\sigma)\| \le Kh_\sigma^{k_1}\|\sigma\|_{k_1+1}.\end{cases} \tag{3.1}$$

For $w \in H^1(\Omega)$, $w|_\Gamma = 0$, we define its elliptic project $\mathrm{R}w \in S_{h_u}$ such that

$$(\nabla(\mathrm{R}w - w), \nabla v_h) + \lambda(\mathrm{R}w - w, v_h) = 0, \quad \forall v_h \in S_{h_u}, \tag{3.2}$$

where $\lambda > 0$ is a positive constant. It is clear there hold the standard error estimates [18]

$$\begin{cases}\|w^n - \mathrm{R}w^n\|_s \le Kh_u^{l+1-s}\|w\|_{L^\infty(H^{l+1}(\Omega))}, \quad s = 0, 1,\\ \|D_t(w^n - \mathrm{R}w^n)\|_s \le Kh_u^{l+1-s}[\|w\|_{L^\infty(H^{l+1}(\Omega))} + \|w_t\|_{L^\infty(H^{l+1}(\Omega))}].\end{cases} \tag{3.3}$$

Now we consider the error estimate for Scheme **I**. Letting $\psi = 0$ in (2.8) and using Lemma 2.1, we have

$$(cw^n, v) + \Delta t(A\nabla w^n, \nabla v) = (w^{n-1} + \Delta t f^n + \Delta t R_1^n, v) + \Delta t(\sigma^{n-1} + \nabla w^{n-1} + \Delta t R_2^n, \nabla v), \quad \forall v \in S. \tag{3.4}$$

Subtracting (2.21) in Remark 2.1 from (3.4), we obtain

$$\big(c(w^n - w_h^n), v_h\big) + \Delta t\,\big(A\nabla(w^n - w_h^n), \nabla v_h\big) = \big(w^{n-1} - w_h^{n-1} + \Delta t R_1^n, v_h\big)$$
$$+\,\Delta t(\sigma^{n-1} - \sigma_h^{n-1} + \nabla(w^{n-1} - w_h^{n-1}) + \Delta t R_2^n, \nabla v_h), \quad \forall v_h \in S_{h_u}. \tag{3.5}$$

Set $\theta^n = \mathrm{R}w^n - w_h^n$, $\rho^n = \mathrm{R}w^n - w^n$, $\pi^n = \mathrm{Q}\sigma^n - \sigma_h^n$ and $\epsilon^n = \mathrm{Q}\sigma^n - \sigma^n$. The estimates for ϵ^n and ρ^n were given in (3.1) and (3.3). We need to estimate θ^n and π^n. From (3.5) we see that θ^n satisfies the following error equation

$$\big(c(\theta^n - \rho^n), v_h\big) + \Delta t\,\big(A\nabla(\theta^n - \rho^n), \nabla v_h\big) = \big(\theta^{n-1} - \rho^{n-1} + \Delta t R_1^n, v_h\big)$$
$$+\,\Delta t(\pi^{n-1} - \epsilon^{n-1} + \nabla(\theta^{n-1} - \rho^{n-1}) + \Delta t R_2^n, \nabla v_h), \quad \forall v_h \in S_{h_u}. \tag{3.6}$$

Equivalently

$$(\theta^n - \theta^{n-1}, v_h) + \Delta t\,\big(\nabla(\theta^n - \theta^{n-1}), \nabla v_h\big) = \Delta t(D_t\rho^n, v_h) - \Delta t(q(\theta^n - \rho^n), v_h)$$
$$-\,\lambda\Delta t(\rho^n - \rho^{n-1}, v_h) - \Delta t^2(\nabla\theta^n, \nabla v_h) - \lambda\Delta t^2(\rho^n, v_h) + \Delta t(\pi^{n-1} - \epsilon^{n-1}, \nabla v_h)$$
$$+\,\Delta t(R_1^n, v_h) + \Delta t^2(R_2^n, \nabla v_h), \quad \forall v_h \in S_{h_u}, \tag{3.7}$$

where we have used the fact that

$$(\nabla\rho^n, \nabla v_h) + \lambda(\rho^n, v_h) = 0.$$

Setting $v_h = \theta^n$ and using Cauchy's inequality, we have that

$$\|\theta^n\|^2 + \Delta t\|\nabla\theta^n\|^2 \le \|\theta^{n-1}\|^2 + \Delta t\|\nabla\theta^{n-1}\|^2 + K\Delta t\,\big(\|\theta^n\|^2 + \|D_t\rho^n\|^2 + \|\rho^n\|^2 + \|\rho^{n-1}\|^2 + \Delta t\|\nabla\theta^n\|^2\big)$$
$$+\,K\Delta t\,\big(\|\mathrm{div}\,\epsilon^{n-1}\|^2 + \|R_1^n\|^2 + \|R_2^n\|^2\big) + \delta\Delta t\|\mathrm{div}\,\pi^{n-1}\|^2. \tag{3.8}$$

Summing for n from 1 to J, $J \le N$, we have that

$$\|\theta^J\|^2 + \Delta t\|\nabla\theta^J\|^2 \le K\sum_{n=1}^{J}\Delta t(\|\theta^n\|^2 + \Delta t\|\nabla\theta^n\|^2) + \delta\sum_{n=1}^{J}\Delta t\|\mathrm{div}\,\pi^n\|^2 + K(h_u^{2(l+1)} + h_\sigma^{2k_1} + \Delta t^2), \tag{3.9}$$

where we have used the initial approximation $w_h^0 = Rw^0, \theta^0 = 0$.

Now we estimates π^n. Letting $v = 0$ in (2.8) and using Lemma 2.1, we have

$$(A^{-1}\sigma^n, \psi) + \Delta t(c^{-1}\mathrm{div}\,\sigma^n, \mathrm{div}\,\psi) = (A^{-1}\sigma^{n-1}, \psi) - \Delta t\,\big((c^{-1}q - A^{-1})w^{n-1}, \mathrm{div}\,\psi\big)$$
$$+\,\Delta t(c^{-1}f^n, \mathrm{div}\,\psi) + \Delta t(c^{-1}R_1^n, \mathrm{div}\,\psi) + \Delta t(A^{-1}R_2^n, \psi), \quad \forall\psi \in H. \tag{3.10}$$

Subtracting (2.22) in Remark 2.1 from (3.10), we obtain

$$(A^{-1}(\sigma^n - \sigma_h^n), \psi_h) + \Delta t(c^{-1}\mathrm{div}\,(\sigma^n - \sigma_h^n), \mathrm{div}\,\psi_h) = (A^{-1}(\sigma^{n-1} - \sigma_h^{n-1}), \psi_h)$$
$$-\,\Delta t\,\big((c^{-1}q - A^{-1})(w^{n-1} - w_h^{n-1}), \mathrm{div}\,\psi_h\big) + \Delta t(c^{-1}R_1^n, \mathrm{div}\,\psi_h) + \Delta t(A^{-1}R_2^n, \psi_h), \quad \forall\psi_h \in H_{h_\sigma}. \tag{3.11}$$

We see that π^n satisfies the following error equation

$$(A^{-1}(\pi^n - \pi^{n-1}), \psi_h) + \Delta t(c^{-1}\mathrm{div}\,\pi^n, \mathrm{div}\,\psi_h) = \Delta t(A^{-1}D_t\epsilon^n, \psi_h) + \Delta t(c^{-1}\mathrm{div}\,\epsilon^n, \mathrm{div}\,\psi_h) + \Delta t(c^{-1}R_1^n, \mathrm{div}\,\psi_h)$$
$$-\,\Delta t\,\big((c^{-1}q - A^{-1})(\theta^{n-1} - \rho^{n-1}), \mathrm{div}\,\psi_h\big) + \Delta t(A^{-1}R_2^n, \psi_h), \quad \forall\psi_h \in H_{h_\sigma}. \tag{3.12}$$

Setting $\psi_h = \pi^n$ and using Cauchy's inequality, we have that

$$\|A^{-\frac{1}{2}}\pi^n\|^2 + \Delta t\|c^{-\frac{1}{2}}\mathrm{div}\,\pi^n\|^2 \le \|A^{-\frac{1}{2}}\pi^{n-1}\|^2 + K\Delta t(\|A^{-\frac{1}{2}}\pi^n\|^2 + \|\theta^{n-1}\|^2 + \|D_t\epsilon^n\|^2 + \|\mathrm{div}\,\epsilon^n\|^2)$$
$$+\,K\Delta t(\|\rho^{n-1}\|^2 + \|R_1^n\|^2 + \|R_2^n\|^2) + \delta\Delta t\|c^{-\frac{1}{2}}\mathrm{div}\,\pi^n\|^2. \tag{3.13}$$

Summing for n from 1 to J, $J \le N$, we obtain

$$\|\pi^J\|^2 + \sum_{n=1}^{J}\Delta t\|\mathrm{div}\,\pi^n\|^2 \le K\sum_{n=1}^{J}\Delta t(\|\pi^n\|^2 + \|\theta^n\|^2) + K(h_u^{2(l+1)} + h_\sigma^{2k_1} + \Delta t^2). \tag{3.14}$$

Adding (3.9) and (3.14) together, we get the estimate

$$\|\theta^J\|^2 + \|\pi^J\|^2 + \Delta t\|\nabla\theta^J\|^2 + \sum_{n=1}^{J}\Delta t\|\mathrm{div}\,\pi^n\|^2 \le K\sum_{n=1}^{J}\Delta t(\|\theta^n\|^2 + \|\pi^n\|^2$$
$$+\,\Delta t\|\nabla\theta^n\|^2) + K(h_u^{2(l+1)} + h_\sigma^{2k_1} + \Delta t^2). \tag{3.15}$$

An application of discrete Gronwall's lemma to (3.15) leads to

$$\max_{0\le n\le N}\|\theta^n\|^2+\max_{0\le n\le N}\|\pi^n\|^2+\sum_{n=1}^{J}\Delta t\|\mathrm{div}\,\pi^n\|^2\le K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^2). \tag{3.16}$$

We have finished the estimates for θ^n and π^n. Noticing $w^n-w_h^n=\theta^n-\rho^n$ and $\sigma^n-\sigma_h^n=\pi^n-\epsilon^n$, we have

$$\max_{0\le n\le N}\|w^n-w_h^n\|^2+\max_{0\le n\le N}\|\sigma^n-\sigma_h^n\|^2+\sum_{n=1}^{J}\Delta t\|\mathrm{div}\,(\sigma^n-\sigma_h^n)\|^2\le K\left(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^2\right). \tag{3.17}$$

Since $w=u_t$, we define u_h^n as

$$\begin{cases} w^n=\dfrac{u^n-u^{n-1}}{\Delta t}+R_5^n,\\ w_h^n=\dfrac{u_h^n-u_h^{n-1}}{\Delta t}, \end{cases} \tag{3.18}$$

where

$$R_5^n=D_tu^n-u_t^n=O(\Delta t).$$

Then we have the inequality

$$\begin{aligned}\|u^J-u_h^J\|^2&\le K\sum_{n=1}^{J}\Delta t\|u^n-u_h^n\|^2+K\sum_{n=1}^{J}\Delta t\left\|\frac{u^n-u_h^n-(u^{n-1}-u_h^{n-1})}{\Delta t}\right\|^2+\|u^0-u_h^0\|^2\\ &\le K\sum_{n=1}^{J}\Delta t\|u^n-u_h^n\|^2+K\sum_{n=1}^{J}\Delta t\|w^n-w_h^n\|^2+h_u^{2(l+1)}+\Delta t^2.\end{aligned} \tag{3.19}$$

Since

$$\sum_{n=1}^{J}\Delta t\|w^n-w_h^n\|^2\le K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^2),$$

applying the discrete Gronwall's lemma to (3.19), we derive the estimate

$$\max_{0\le n\le N}\|u^n-u_h^n\|^2\le K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^2). \tag{3.20}$$

We are able to demonstrate our main result for Scheme **I**.

Theorem 3.1. *Let (σ,w) be the solution of system (2.4) and (σ_h^n,w_h^n) be the solution of Scheme **I**. When the solution (σ,w) is sufficiently smooth and Δt, h_u and h_σ are sufficiently small there holds the priori error estimate*

$$\max_{0\le n\le N}\|u^n-u_h^n\|+\max_{0\le n\le N}\|w^n-w_h^n\|+\max_{0\le n\le N}\|\sigma^n-\sigma_h^n\|\le K\{h_\sigma^{k_1}+h_u^{l+1}+\Delta t\}, \tag{3.21}$$

where the constant K is dependent upon T and some norms of the solution (σ,u) and independent of the mesh parameters h_u, h_σ and Δt.

In analogy with the analysis for Scheme **I**, we analyze the convergence of Scheme **II**. Letting $\psi=0$ in (2.13) and using Lemma 2.1, we have

$$\begin{aligned}(\tilde{c}w^n,v)+\frac{\Delta t}{2}(\tilde{A}\nabla w^n,\nabla v)&=\left((\tilde{c}-\Delta tq)w^{n-1}-\frac{\Delta t}{2}\mathrm{div}\,\sigma^{n-1}+\Delta tf^{n-\frac{1}{2}}+\Delta tR_3^n,v\right)\\ &\quad+\frac{\Delta t}{2}\left(\sigma^{n-1}+(\tilde{A}-\Delta t)\nabla w^{n-1}+\Delta tR_4^n,\nabla v\right),\quad\forall v\in S.\end{aligned} \tag{3.22}$$

Subtracting (2.23) in Remark 2.2 from (3.22), we obtain

$$\begin{aligned}&\left(\tilde{c}(w^n-w_h^n),v_h\right)+\frac{\Delta t}{2}\left(\tilde{A}\nabla(w^n-w_h^n),\nabla v_h\right)=\left((\tilde{c}-\Delta tq)(w^{n-1}-w_h^{n-1})+\Delta tR_3^n,v_h\right)+\Delta t(\sigma^{n-1}-\sigma_h^{n-1},\nabla v_h)\\ &\quad+\frac{\Delta t}{2}\left((\tilde{A}-\Delta t)\nabla(w^{n-1}-w_h^{n-1})+\Delta tR_4^n,\nabla v_h\right),\quad\forall v_h\in S_{h_u}.\end{aligned} \tag{3.23}$$

We see that θ^n satisfies the following error equation

$$\begin{aligned}
&(\tilde{c}(\theta^n-\theta^{n-1}),v_h)+\frac{\Delta t}{2}\left(\tilde{A}\nabla(\theta^n-\theta^{n-1}),\nabla v_h\right)=\Delta t(\tilde{c}D_t\rho^n,v_h)-\Delta t(q(\theta^{n-1}-\rho^{n-1}),v_h)\\
&\quad-\lambda\frac{\Delta t}{2}(\tilde{A}(\rho^n-\rho^{n-1}),v_h)-\frac{\Delta t^2}{2}(\nabla\theta^{n-1},\nabla v_h)-\lambda\frac{\Delta t^2}{2}(\rho^{n-1},v_h)\\
&\quad+\Delta t(\pi^{n-1}-\epsilon^{n-1},\nabla v_h)+\Delta t(R_3^n,v_h)+\frac{\Delta t^2}{2}(R_4^n,\nabla v_h),\quad\forall v_h\in S_{h_u}.
\end{aligned}\tag{3.24}$$

Setting $v_h=\theta^n+\theta^{n-1}$, then using the equality

$$\Delta t\left(\pi^{n-1},\nabla(\theta^n+\theta^{n-1})\right)=\Delta t\left[(\pi^{n-1},\nabla\theta^{n-1})-(\pi^n,\nabla\theta^n)-(\operatorname{div}(\pi^n+\pi^{n-1}),\theta^n)\right],$$

and Cauchy's inequality, we have

$$\begin{aligned}
&\|\tilde{c}^{\frac12}\theta^n\|^2+\frac{\Delta t}{2}\|\tilde{A}^{\frac12}\nabla\theta^n\|^2\le\|\tilde{c}^{\frac12}\theta^{n-1}\|^2+\frac{\Delta t}{2}\|\tilde{A}^{\frac12}\nabla\theta^{n-1}\|^2+\Delta t\left[(\pi^{n-1},\nabla\theta^{n-1})-(\pi^n,\nabla\theta^n)\right]\\
&\quad+K\Delta t\left(\|\theta^n\|^2+\|\theta^{n-1}\|^2+\Delta t\|\nabla\theta^n\|^2+\Delta t\|\nabla\theta^{n-1}\|^2+\|\operatorname{div}\epsilon^{n-1}\|^2\right)\\
&\quad+K\Delta t\left(\|D_t\rho^n\|^2+\|\rho^n\|^2+\|\rho^{n-1}\|^2+\|R_3^n\|^2+\|R_4^n\|^2\right)+\delta\Delta t\|\operatorname{div}(\pi^n+\pi^{n-1})\|^2.
\end{aligned}\tag{3.25}$$

Summing for n from 1 to J, $J\le N$, we have that

$$\begin{aligned}
\|\theta^J\|^2+\Delta t\|\nabla\theta^J\|^2&\le K\sum_{n=1}^{J}\Delta t(\|\theta^n\|^2+\Delta t\|\nabla\theta^n\|^2)+\delta\sum_{n=1}^{J}\Delta t\|\operatorname{div}(\pi^n+\pi^{n-1})\|^2\\
&\quad+K\Delta t\|\pi^J\|^2+K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^4),
\end{aligned}\tag{3.26}$$

where we have used the initial approximation $w_h^0=Rw^0,\theta^0=0$.

Now we estimates π^n. Letting $v=0$ in (2.13) and using Lemma 2.1, we have

$$\begin{aligned}
&(\tilde{A}^{-1}\sigma^n,\psi)+\frac{\Delta t}{2}(\tilde{c}^{-1}\operatorname{div}\sigma^n,\operatorname{div}\psi)=(\tilde{A}^{-1}\sigma^{n-1},\psi)-\frac{\Delta t}{2}(\tilde{c}^{-1}\operatorname{div}\sigma^{n-1},\operatorname{div}\psi)-\Delta t\left((\tilde{c}^{-1}q-\tilde{A}^{-1})w^{n-1},\operatorname{div}\psi\right)\\
&\quad+\Delta t(\tilde{c}^{-1}f^{n-\frac12},\operatorname{div}\psi)+\Delta t(\tilde{c}^{-1}R_3^n,\operatorname{div}\psi)+\Delta t(\tilde{A}^{-1}R_4^n,\psi),\quad\forall\psi\in H.
\end{aligned}\tag{3.27}$$

Subtracting (2.24) in Remark 2.2 from (3.27), we see that π^n satisfies the following error equation

$$\begin{aligned}
&(\tilde{A}^{-1}(\pi^n-\pi^{n-1}),\psi_h)+\frac{\Delta t}{2}\left(\tilde{c}^{-1}\operatorname{div}(\pi^n+\pi^{n-1}),\operatorname{div}\psi_h\right)=\Delta t(\tilde{A}^{-1}D_t\epsilon^n,\psi_h)+\frac{\Delta t}{2}\left(\tilde{c}^{-1}\operatorname{div}(\epsilon^n+\epsilon^{n-1}),\operatorname{div}\psi_h\right)\\
&\quad+\Delta t(\tilde{c}^{-1}R_3^n,\operatorname{div}\psi_h)-\Delta t\left((\tilde{c}^{-1}q-\tilde{A}^{-1})(\theta^{n-1}-\rho^{n-1}),\operatorname{div}\psi_h\right)+\Delta t(\tilde{A}^{-1}R_4^n,\psi_h),\quad\forall\psi_h\in H_{h_\sigma}.
\end{aligned}\tag{3.28}$$

Setting $\psi_h=\pi^n+\pi^{n-1}$ and using Cauchy's inequality, we have that

$$\begin{aligned}
\|\tilde{A}^{-\frac12}\pi^n\|^2+\frac{\Delta t}{2}\|\tilde{c}^{-\frac12}\operatorname{div}(\pi^n+\pi^{n-1})\|^2&\le\|\tilde{A}^{-\frac12}\pi^{n-1}\|^2+K\Delta t(\|\tilde{A}^{-\frac12}\pi^n\|^2+\|\tilde{A}^{-\frac12}\pi^{n-1}\|^2+\|\theta^{n-1}\|^2)\\
&\quad+K\Delta t(\|\rho^{n-1}\|^2+\|D_t\epsilon^n\|^2+\|\operatorname{div}\epsilon^n\|^2+\|\operatorname{div}\epsilon^{n-1}\|^2)\\
&\quad+K\Delta t(\|R_3^n\|^2+\|R_4^n\|^2)+\delta\Delta t\|\tilde{c}^{-\frac12}\operatorname{div}(\pi^n+\pi^{n-1})\|^2.
\end{aligned}\tag{3.29}$$

Summing for n from 1 to J, $J\le N$, we obtain

$$\|\pi^J\|^2+\sum_{n=1}^{J}\Delta t\|\operatorname{div}(\pi^n+\pi^{n-1})\|^2\le K\sum_{n=1}^{J}\Delta t(\|\pi^n\|^2+\|\theta^n\|^2)+K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^4).\tag{3.30}$$

Adding (3.26) and (3.30) together and applying the discrete Gronwall's lemma, we drive the estimate

$$\max_{0\le n\le N}\|\theta^n\|^2+\max_{0\le n\le N}\|\pi^n\|^2+\sum_{n=1}^{J}\Delta t\|\operatorname{div}(\pi^n+\pi^{n-1})\|^2\le K(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^4).\tag{3.31}$$

We have finished the estimates for θ^n and π^n for Scheme **II**. We get

$$\max_{0\le n\le N}\|w^n-w_h^n\|^2+\max_{0\le n\le N}\|\sigma^n-\sigma_h^n\|^2+\sum_{n=1}^{J}\Delta t\|\operatorname{div}(\hat{\sigma}^{n-\frac12}-\hat{\sigma}_h^{n-\frac12})\|^2\le K\left(h_u^{2(l+1)}+h_\sigma^{2k_1}+\Delta t^4\right).\tag{3.32}$$

Since $w = u_t$, we define u_h^n as following

$$\begin{cases} w^n = \dfrac{\hat{u}^{n+\frac{1}{2}} - \hat{u}^{n-\frac{1}{2}}}{\Delta t} + R_6^n, \\ w_h^n = \dfrac{\hat{u}_h^{n+\frac{1}{2}} - \hat{u}_h^{n-\frac{1}{2}}}{\Delta t}, \end{cases} \tag{3.33}$$

where

$$\hat{u}^{n+\frac{1}{2}} = \frac{u^n + u^{n-1}}{2},$$

$$R_6^n = \frac{\hat{u}^{n+\frac{1}{2}} - \hat{u}^{n-\frac{1}{2}}}{\Delta t} - u_t^n = O(\Delta t^2).$$

Define u_h^1 as

$$u_h^1 = \Delta t w_0(x) + u_h^0, \tag{3.34}$$

then

$$\|u^1 - u_h^1\| \le K(h_u^{l+1} + \Delta t^2). \tag{3.35}$$

Then we have the inequality

$$\begin{aligned} \|\hat{u}^{J-\frac{1}{2}} - \hat{u}_h^{J-\frac{1}{2}}\|^2 &\le K\sum_{n=1}^{J} \Delta t \|\hat{u}^{n-\frac{1}{2}} - \hat{u}_h^{n-\frac{1}{2}}\|^2 + K\sum_{n=1}^{J} \Delta t \left\| \frac{\hat{u}^{n-\frac{1}{2}} - \hat{u}_h^{n-\frac{1}{2}} - (\hat{u}^{n-\frac{3}{2}} - \hat{u}_h^{n-\frac{3}{2}})}{\Delta t} \right\|^2 + \|\hat{u}^{\frac{1}{2}} - \hat{u}_h^{\frac{1}{2}}\|^2 \\ &\le K\sum_{n=1}^{J} \Delta t \|\hat{u}^{n-\frac{1}{2}} - \hat{u}_h^{n-\frac{1}{2}}\|^2 + K\sum_{n=1}^{J} \Delta t \|w^n - w_h^n\|^2 + h_u^{2(l+1)} + \Delta t^4. \end{aligned} \tag{3.36}$$

Since

$$\sum_{n=1}^{J} \Delta t \|w^n - w_h^n\|^2 \le K(h_u^{2(l+1)} + h_\sigma^{2k_1} + \Delta t^4),$$

by virtue of Gronwall's inequality, we derive the estimate

$$\max_{0\le n\le N} \|u^n - u_h^n\|^2 \le K(h_u^{2(l+1)} + h_\sigma^{2k_1} + \Delta t^4). \tag{3.37}$$

We are able to demonstrate our main result for Scheme **II**.

Theorem 3.2. *Let (σ, w) be the solution of system (2.4) and (σ_h^n, w_h^n) be the solution of Scheme **II.** When the solution (σ, w) is sufficiently smooth and Δt, h_u and h_σ are sufficiently small there holds the priori error estimate*

$$\max_{0\le n\le N} \|u^n - u_h^n\| + \max_{0\le n\le N} \|w^n - w_h^n\| + \max_{0\le n\le N} \|\sigma^n - \sigma_h^n\| \le K(h_\sigma^{k_1} + h_u^{l+1} + \Delta t^2), \tag{3.38}$$

where the constant K is dependent upon T and some norms of the solution (σ, u) and independent of the mesh parameters h_u, h_σ and Δt.

4. Numerical example

In this section, we give a numerical example to show the efficiency of the presented schemes.
We consider the following problem in a two-dimensional domain

$$\begin{cases} u_{tt} - \Delta u_t - \Delta u + q u_t = f(x, t), \quad x \in \Omega,\ 0 < t \le T, \\ u(x, t) = 0, \qquad u_t(x, t) = 0, \quad x \in \Gamma,\ 0 < t \le T, \\ u(x, 0) = \sin x \sin y, \quad x \in \Omega, \\ u_t(x, 0) = \dfrac{1}{2} \sin x \sin y, \quad x \in \Omega, \end{cases} \tag{4.1}$$

where $\Omega = (0, \pi] \times (0, \pi]$, q is a constant, $f(x, t) = (\frac{13}{4} + \frac{q}{2})e^{\frac{t}{2}} \sin x \sin y$. The analytical solution of (4.1) is $u = e^{\frac{t}{2}} \sin x \sin y$. $w = u_t = \frac{1}{2} e^{\frac{t}{2}} \sin x \sin y$. Then

$$\sigma = -(\nabla u_t + \nabla u) = \begin{pmatrix} -\dfrac{3}{2} e^{\frac{t}{2}} \cos x \sin y \\ -\dfrac{3}{2} e^{\frac{t}{2}} \sin x \cos y \end{pmatrix}.$$

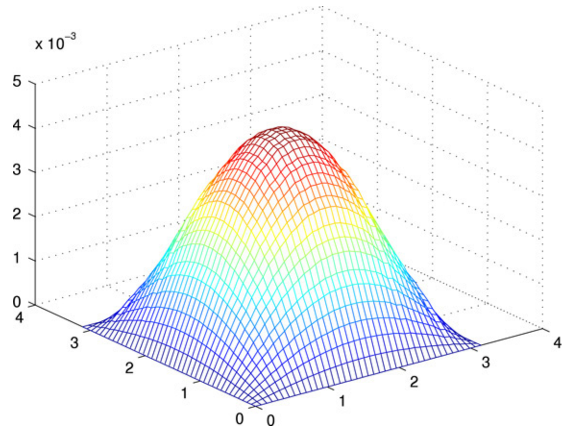

Fig. 1. The error $|u - u_h|$ of scheme **I**: $q = 1.0$, $T = 1$, $\Delta t = \frac{1}{100}$, $h = \frac{\pi}{40}$.

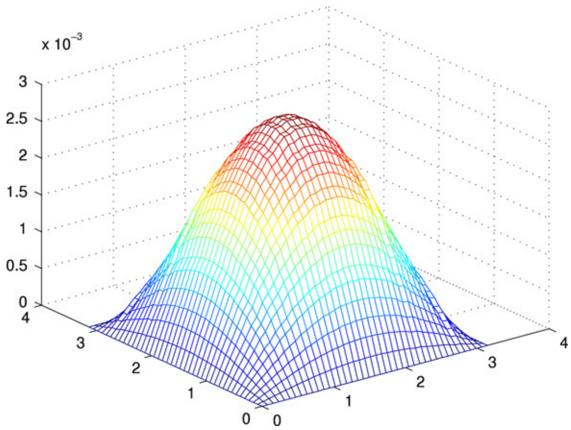

Fig. 2. The error $|u - u_h|$ of scheme **II:** $q = 1.0$, $T = 1$, $\Delta t = \frac{1}{20}$, $h = \frac{\pi}{40}$.

Table 1
The numerical results of Scheme **I** with $q = 100.0$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{25}, \frac{\pi}{10})$	5.62e−002	1.36e−002	4.50e−002	8.83e−002	2.14e−002	1.10e−001
$(\frac{1}{50}, \frac{\pi}{20})$	1.55e−002	3.38e−003	1.22e−002	2.43e−002	5.30e−003	2.84e−002
$(\frac{1}{100}, \frac{\pi}{40})$	4.64e−003	8.33e−004	4.10e−003	7.31e−003	1.31e−003	8.48e−003
Order	1.80	2.02	1.73	1.80	2.01	1.85

We divide each direction into uniform intervals and consider the bilinear finite element defined on uniform rectangles. Eu denotes the error estimate about u, Ew denotes the error estimate about w and $E\sigma$ denotes the error estimate about σ. L^∞ denote the maximum error, and L^2 denotes the discrete L^2 norm. For a set of simulations, different mesh sizes and different values of q are taken. The results of the schemes are as follows ($T = 1$). Tables 1–3 and Fig. 1 list the numerical results of Scheme **I**, and Tables 4–6 and Fig. 2 list the numerical results of Scheme **II.**

From the numerical results, we can obtain the following conclusions.

1. The numerical results are consistent with the theoretical analysis. Even for smaller q-value, the numerical results are very good. The two split least-squares Galerkin methods are reasonable.

2. Compared with Scheme **I**, Scheme **II** improves the precision in time. So we can adopt larger time steps with no loss of accuracy.

3. From the numerical results, we conclude that the methods also yield approximate solutions with optimal accuracy in $(L^\infty(\Omega))^2 \times L^\infty(\Omega)$ norms. Theoretical analysis will be considered in a forthcoming paper.

Table 2
The numerical results of Scheme **I** with $q = 1.0$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{25}, \frac{\pi}{10})$	5.77e−002	1.71e−002	3.81e−002	9.06e−002	2.68e−002	9.26e−002
$(\frac{1}{50}, \frac{\pi}{20})$	1.58e−002	3.97e−003	1.06e−002	2.48e−002	6.24e−003	2.47e−002
$(\frac{1}{100}, \frac{\pi}{40})$	4.66e−003	7.57e−004	3.31e−003	7.36e−003	1.23e−003	7.32e−003
Order	1.82	2.25	1.76	1.81	2.22	1.83

Table 3
The numerical results of Scheme **I** with $q = 0.01$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{25}, \frac{\pi}{10})$	5.80e−002	1.80e−002	3.89e−002	9.11e−002	2.83e−002	9.46e−002
$(\frac{1}{50}, \frac{\pi}{20})$	1.59e−002	4.16e−003	1.08e−002	2.49e−002	6.53e−003	2.51e−002
$(\frac{1}{100}, \frac{\pi}{40})$	4.69e−003	7.98e−004	3.24e−003	7.40e−003	1.28e−003	7.31e−003
Order	1.81	2.25	1.79	1.81	2.23	1.85

Table 4
The numerical results of Scheme **II** with $q = 100.0$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{5}, \frac{\pi}{10})$	4.45e−002	1.23e−002	3.83e−002	6.98e−002	1.93e−002	9.33e−002
$(\frac{1}{10}, \frac{\pi}{20})$	1.17e−002	3.14e−003	9.27e−003	1.83e−002	4.93e−003	2.16e−002
$(\frac{1}{20}, \frac{\pi}{40})$	2.90e−003	7.75e−004	2.53e−003	4.57e−003	1.23e−003	5.23e−003
Order	1.97	1.99	1.96	1.97	1.99	2.08

Table 5
The numerical results of Scheme **II** with $q = 1.0$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{5}, \frac{\pi}{10})$	4.41e−002	1.46e−002	3.12e−002	6.92e−002	2.30e−002	7.59e−002
$(\frac{1}{10}, \frac{\pi}{20})$	1.19e−002	4.02e−003	7.74e−003	1.87e−002	6.32e−003	1.80e−002
$(\frac{1}{20}, \frac{\pi}{40})$	2.97e−003	1.03e−003	1.96e−003	4.70e−003	1.64e−003	4.37e−003
Order	1.94	1.91	2.00	1.94	1.90	2.06

Table 6
The numerical results of Scheme **II** with $q = 0.01$.

$(\Delta t, h)$	L^∞			L^2		
	Eu	Ew	$E\sigma$	Eu	Ew	$E\sigma$
$(\frac{1}{5}, \frac{\pi}{10})$	4.41e−002	1.51e−002	3.08e−002	6.92e−002	2.38e−002	7.50e−002
$(\frac{1}{10}, \frac{\pi}{20})$	1.20e−002	4.24e−003	7.85e−003	1.88e−002	6.66e−003	1.83e−002
$(\frac{1}{20}, \frac{\pi}{40})$	2.99e−003	1.09e−003	2.02e−003	4.72e−003	1.73e−003	4.51e−003
Order	1.94	1.90	1.97	1.94	1.89	2.03

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] R. Arima, Y. Hasegawa, On global solutions for mixed problems of a semi-linear differential equation, Proc. Japan Acad. 39 (1963) 721–725.
[3] I. Babuśka, The finite element method with lagrange multipliers, Numer. Math. 20 (1973) 179–192.
[4] F. Brezzi, On the existence uniqueness and approximation of saddle point problems arising from Lagrange multipliers, RAIRO Sér. Anal. Numér. 8 (1974) 129–151.
[5] P.G. Ciarlet, Finite Element Methods for Elliptic Problems, North-Holland, New York, 1978.
[6] Hui Guo, Hongxing Rui, Least-squares Galerkin procedures for pseudohyperbolic equations, Appl. Math. Computat. 189 (2007) 425–439.
[7] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flows, Gordon and Breach, London, 1969.
[8] J. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon, Proc. IRE 50 (1962) 91–102.
[9] J.C. Nedelec, Mixed finite element in R^3, Numer. Math. 35 (1980) 315–341.

[10] C.V. Pao, A mixed initial boundary value problem arising in Neurophysiology, J. Math. Anal. Appl. 52 (1975) 105–119.
[11] C.V. Pao, Positive solution of a nonlinear diffusion system arising in chemical reactors, J. Math. Anal. Appl. 46 (1974) 820–835.
[12] C.V. Pao, Successive approximation of some nonlinear initial boundary-value problems, SIAM J. Math. Anal. 5 (1974) 91–102.
[13] G. Ponce, Global existence of small of solutions to a class of nonlinear evolution equations, Nonlinear Anal. 9 (1985) 399–418.
[14] P.A. Raviart, J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, in: Mathematical Aspects of Finite Element Methods, in: Lecture Notes in Mathematics, vol. 606, Springer, Berlin, 1977, pp. 292–315.
[15] Hongxing Rui, S.D. Kim, S. Kim, Split least-squares finite element methods for linear and nonlinear parabolic problems, J. Comp. Appl. Math., in press (doi:10.1016/j.cam.2008.03.030).
[16] Hongxing Rui, Seokchan Kim, Sang Dong Kim, A remark on least-squares mixed element methods for reaction–diffusion problems, J. Comp. Appl. Math. 202 (2007) 230–236.
[17] Weiming Wan, Yacheng Liu, Long time behaviors of solutions for initial boundary value problem of pseudohyperbolic equations, Acta Math. Appl. Sinica 22 (1999) 311–355.
[18] M.F. Wheeler, A priori L^2 error estimates for Galerkin approximation to parabolic partial differential equations, SIAM Numer. Anal. 10 (1973) 723–759.